AROUND A SOBOLEV–ORLICZ INEQUALITY FOR OPERATORS OF GIVEN SPECTRAL DENSITY

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Abstract. We prove some general Sobolev–Orlicz, Nash and Faber–Krahn inequalities for positive operators A of given ultracontractive spectral decay $F(\lambda) = \|\chi_A(]0,\lambda])\|_{1,\infty}$. For invariant operators on coverings of finite simplicial complexes this function is equivalent to von Neumann spectral density. This allows in the polynomial decay case to relate the Novikov–Shubin numbers to Sobolev inequalities on exact ℓ^2-cochains, and to the vanishing of the torsion of the $\ell^{p,2}$-cohomology for some $p \geq 2$.

1. Introduction and main results

Let A be a strictly positive self-adjoint operator on a measure space (X, μ). Suppose moreover that the semigroup e^{-tA} is equicontinuous on $L^1(X)$. Then, according to Varopoulos [15, 6], a polynomial heat decay

$$\|e^{-tA}\|_{1,\infty} \leq Ct^{-\alpha/2} \quad \text{with} \quad \alpha > 2\,,$$

is equivalent to the Sobolev inequality

$$\|f\|_p \leq C'\|A^{1/2}f\|_2 \quad \text{for} \quad 1/p = 1/2 - 1/\alpha. \tag{1}$$

This result applies in particular in the case A is the Laplacian acting *on scalar functions* of a complete manifold, either in the smooth or discrete graph setting.

The first purpose of this paper is to present short proofs of general Sobolev–Orlicz inequalities that hold for positive self-adjoint operators, without equicontinuity or polynomial decay assumption, knowing either their heat decay, as previously, or their "ultracontractive spectral decay" $F(\lambda) = \|\Pi_\lambda\|_{1,\infty}$ of their spectral projectors $\Pi_\lambda = \chi_A(]0,\lambda])$ on E_λ. As will be seen in Sections 4 and 5, the interest for this former $F(\lambda)$ mostly comes from geometric considerations. For instance if A is a scalar invariant operator over a discrete group Γ, or more generally an unimodular one, then $F(\lambda)$ coincides with von Neumann's Γ-dimension of E_λ, and thus F represents the spectral density function of A, see Proposition 4.2. In the general setting the spectral decay F stays a right continuous increasing function as comes from the identity

$$\|P^*P\|_{1,\infty} = \|P\|_{1,2}^2 = \sup_{\|f\|_1,\|g\|_1\leq 1} |\langle Pf, Pg\rangle|. \tag{2}$$

Date: February 16, 2009.

2000 *Mathematics Subject Classification.* 58J50, 46E35, 58J35, 46E30, 35P20.

Key words and phrases. Sobolev–Orlicz inequality, spectral distribution, Faber–Krahn inequality, Novikov–Shubin invariants, $\ell^{p,q}$-cohomology.

Author supported in part by the French ANR-06-BLAN60154-01 grant.

We state the Sobolev–Orlicz inequalities we shall prove. In the sequel, if φ is a monotonic function, φ^{-1} will denote its right continuous inverse.

Theorem 1.1. *Let A be a positive self-adjoint operator on (X, μ) with ultracontractive spectral projections $\Pi_\lambda = \chi_A(]0, \lambda])$, i.e. $F(\lambda) = \|\Pi_\lambda\|_{1,\infty} < +\infty$.*

Suppose moreover that the Stieljes integral $G(\lambda) = \displaystyle\int_0^\lambda \frac{dF(u)}{u}$ converges. Then any non zero $f \in L^2(X) \cap (\ker A)^\perp$ of finite energy $\mathcal{E}(f) = \langle Af, f\rangle_2$ satisfies

$$\int_X H\Big(\frac{|f(x)|^2}{4\mathcal{E}(f)}\Big) d\mu \leq 1\,, \tag{3}$$

where $H(y) = y\, G^{-1}(y)$.

The heat version of this result has a similar statement (and proof).

Theorem 1.2. *Let A be a positive self-adjoint operator on (X, μ) such that $L(t) = \|e^{-tA}\Pi_V\|_{1,\infty}$ is finite, with $V = L^2(X) \cap (\ker A)^\perp$.*

Suppose moreover that $M(t) = \displaystyle\int_t^{+\infty} L(u) du < +\infty$. Then any non zero $f \in V$ of finite energy satisfies

$$\int_X N\Big(\frac{|f(x)|^2}{4\mathcal{E}(f)}\Big) d\mu \leq \ln 2\,, \tag{4}$$

where $N(y) = y/M^{-1}(y)$

Both results give (effective) Sobolev inequalities (1) in the polynomial decay case for F or L. At first, one sees easily that the transform from F to G is increasing, see (13), while G to H is decreasing. Therefore, if $F(\lambda) \leq C\lambda^\alpha$ for $\alpha > 1$, then $G(\lambda) \leq C_1 \lambda^{\alpha-1}$ with $C_1 = \frac{C\alpha}{\alpha-1}$, and $H(y) \geq C_1^{\frac{1}{1-\alpha}} y^{\frac{\alpha}{\alpha-1}}$. Hence (3) reads $\|f\|_{2\alpha/(\alpha-1)} \leq 2C_1^{\frac{1}{2\alpha}} \|A^{1/2} f\|_2$.

Under convexity assumptions, H and N-Sobolev inequalities (3) and (4) imply some general Nash and Faber–Krahn inequalities, see (18) and (19). This approach assumes some thinness of the near-zero spectrum, as required by the convergence of G or M. Since the classical Nash inequality makes sense for thick spectrum, one may look for a direct proof. From heat decay to Nash, such a derivation has already been obtained for general operators by Coulhon, see [6] and the survey [7]. Therefore we will focus here on the relationship between the spectral density F and Nash. This states as follows.

Theorem 1.3. *Let A be a positive self-adjoint operator, with finite $F(\lambda) = \|\Pi_\lambda\|_{1,\infty}$, and a non zero $f \in V = L^2(X) \cap (\ker A)^\perp$.*

- *Then it holds that*

$$\int_X |f(x)|^2 F^{-1}\Big(\frac{|f(x)|}{2\|f\|_1}\Big) d\mu \leq 4\mathcal{E}(f)\,. \tag{5}$$

- *If φ is a convex function such that $0 \leq \varphi(y) \leq yF^{-1}(y)$, then the Nash–type inequality holds*

$$\|f\|_1^2 \varphi\Big(\frac{\|f\|_2^2}{2\|f\|_1^2}\Big) \leq 2\mathcal{E}(f)\,. \tag{6}$$

• *In particular if f and Af are supported in a domain Ω of finite measure, the Faber–Krahn type inequality is satisfied*

$$\mu(\Omega)\varphi\Big(\frac{1}{2\mu(\Omega)}\Big) \leq \frac{2\mathcal{E}(f)}{\|f\|_2^2}\,. \tag{7}$$

As an illustration, if one can take above $\varphi(y) \geq CyF^{-1}(y)$ for some constant C, for instance $\varphi(y) = yF^{-1}(y)$ if convex itself, then (7) shows that a non zero state $f \in V$ of energy $\mathcal{E}(f) \leq \lambda\|f\|_2^2$ and support Ω satisfies the simple uncertainty principle

$$2\mu(\Omega)F(4\lambda/C) \geq 1\,. \tag{8}$$

On groups this fits well with the interpretation of $F(\lambda)$ as a renormalised "density" of dimension of E_λ per volume. More concretely, for any invariant positive scalar operator on a finite group Γ, one has by Proposition 4.2 that $F(\lambda) = \dim E_\lambda/\mathrm{card}(\Gamma)$, and thus (8) reads

$$2\dim E_{4\lambda/C} \geq \mathrm{card}(\Gamma)/\mathrm{card}(\Omega)\,.$$

Except for the multiplicative constants 2 and $4/C$ this formula is quite sharp in general. Indeed it could happen in some case that Γ be tiled by $N = \mathrm{card}(\Gamma)/\mathrm{card}(\Omega)$ copies of such domains Ω, implying by min-max principle that $\dim E_\lambda \geq N$ there.

The Sobolev-like inequalities in Theorem 1.1 and 1.2 are not restricted to scalar functions and apply in particular to the following setting. Let K be a finite simplicial complex and $X \to K = X/\Gamma$ some covering. One considers on X the complex of ℓ^2 k-cochains with the discrete coboundary

$$d_k : \ell^2X^k \to \ell^2X^{k+1}$$

dual to the usual boundary ∂ of simplexes, see e.g. [13, §3].

Its (ℓ^2) cohomology $H_2^{k+1} = \ker d_{k+1}/\operatorname{Im} d_k$ splits in two components :

- the reduced part $\overline{H}_2^{k+1} = \ker d_{k+1}/\overline{\operatorname{Im} d_k}$, isomorphic to ℓ^2-harmonic cochains $\mathcal{H}_2^{k+1} = \ker d_{k+1} \cap \ker d_k^*$,
- and the torsion $T_2^{k+1} = \overline{\operatorname{Im} d_k}/\operatorname{Im} d_k$.

Although this torsion is not a normed space, one can study it by "measuring" the unboundedness of d_k^{-1} on $\operatorname{Im} d_k$. We will consider here two different means.

- A first one is inspired by $\ell^{p,q}$-cohomology. One enlarges the space ℓ^2X^k to ℓ^pX^k for $p \geq 2$, and asks whether, for p large enough, one has

$$\overline{d_k(\ell^2X^k)}^{\ell^2} \subset d_k(\ell^pX^k)\,, \tag{9}$$

This is satisfied in case the following Sobolev identity holds

$$\exists C \quad \text{such that} \quad \|\alpha\|_p \leq C\|d_k\alpha\|_2 \quad \text{for all } \alpha \in (\ker d_k)^\perp \subset \ell^2\,. \tag{10}$$

The geometric interest of the rougher formulation (9) lies in its stability under the change of X into other bounded homotopy equivalent spaces, as stated in Proposition 5.2. Moreover if $\overline{H}_2^{k+1}(X)$ vanishes, then (9) is equivalent to the vanishing of the torsion of the $\ell^{p,2}$-cohomology of X, as will be seen in Section 5.

- The second approach is spectral and relies on von Neumann Γ-dimension. Consider the Γ-invariant self-adjoint $A = d_k^*d_k$ acting on $(\ker d_k)^\perp$ and the spectral density $F_{\Gamma,k}(\lambda) = \dim_\Gamma E_\lambda$

of its spectral spaces E_λ. This function vanishes near zero if and only if zero is isolated in the spectrum of A, which is equivalent to the vanishing of the torsion T_2^{k+1}. The asymptotic behaviour of $F_{\Gamma,k}(\lambda)$ when $\lambda \searrow 0$ has a geometric interest in general since, given Γ, it is an homotopy invariant of the quotient space K, as shown by Efremov, Gromov and Shubin in [9, 12, 11].

One can compare these two notions in the spirit of Varopoulos result (1) on functions. In the case of polynomial decay one obtains.

Theorem 1.4. *Let K be a finite simplicial space and $X \to K = X/\Gamma$ a covering. Let $F_{\Gamma,k}(\lambda) = \dim_\Gamma E_\lambda$ denotes the spectral density function of $A = d_k^* d_k$ on $(\ker d_k)^\perp$.*

If $F_{\Gamma,k}(\lambda) \leq C\lambda^{\alpha/2}$ for some $\alpha > 2$, then the Sobolev inequality (10), and the inclusion (9), hold for $1/p \leq 1/2 - 1/\alpha$.

If moreover the reduced ℓ^2-cohomology $\overline{H}_2^{k+1}(X)$ vanishes, this implies the vanishing of the $\ell^{p,2}$-torsion of X, as stated in Corollary 5.4.

Other spectral decays than polynomial can be handled with Theorem 1.1, leading then to a bounded inverse of d_k from $\operatorname{Im} d_k \cap \ell^2$ into a more general Orlicz space given by H.

2. Proofs of main inequalities

The first step towards Theorems 1.1 and 1.2 is to consider the ultracontrativity of the auxiliary operators $A^{-1}\Pi_\lambda$ and $A^{-1}e^{-tA}\Pi_V$.

Proposition 2.1. • *Let A, F and G be given as in Theorem 1.1. Then $A^{-1}\Pi_\lambda$ is ultracontractive with*

$$\|A^{-1}\Pi_\lambda\|_{1,\infty} \leq G(\lambda) = \int_0^\lambda \frac{dF(u)}{u}\,. \tag{11}$$

• *Let A, L and M be given as in Theorem 1.2. Then $A^{-1}e^{-tA}\Pi_V$ is ultracontractive with*

$$\|A^{-1}e^{-tA}\Pi_V\|_{1,\infty} \leq M(t) = \int_t^{+\infty} L(s)ds\,. \tag{12}$$

Proof. • The spectral calculus gives

$$A^{-1}(\Pi_\lambda - \Pi_\varepsilon) = \int_{]\varepsilon,\lambda]} u^{-1} d\Pi_u = \lambda^{-1}\Pi_\lambda - \varepsilon^{-1}\Pi_\varepsilon + \int_{]\varepsilon,\lambda]} u^{-2}\Pi_u du\,,$$

thus taking norms, one obtains

$$\begin{aligned}\|A^{-1}(\Pi_\lambda - \Pi_\varepsilon)\|_{1,\infty} &\leq \lambda^{-1}F(\lambda) + \varepsilon^{-1}F(\varepsilon) + \int_{]\varepsilon,\lambda]} u^{-2}F(u)du \\ &= G(\lambda) - G(\varepsilon) + 2\varepsilon^{-1}F(\varepsilon)\,.\end{aligned}$$

Now by finiteness of G, one has $\|\Pi_\varepsilon/\varepsilon\|_{1,\infty} = F(\varepsilon)/\varepsilon \leq G(\varepsilon) \to 0$ when $\varepsilon \searrow 0$, hence by (2)

$$\begin{aligned}\|A^{-1}\Pi_\lambda\|_{1,\infty} &= \|\Pi_\lambda A^{-1/2}\Pi_\lambda\|_{1,2}^2 \\ &= \lim_{\varepsilon\to 0} \|(\Pi_\lambda - \Pi_\varepsilon)A^{-1/2}\Pi_\lambda\|_{1,2}^2 \quad \text{by Beppo-Levi,} \\ &= \lim_{\varepsilon\to 0} \|A^{-1}(\Pi_\lambda - \Pi_\varepsilon)\|_{1,\infty} \leq G(\lambda)\,.\end{aligned}$$

We note that we also have

$$G(\lambda) = \lambda^{-1}F(\lambda) + \int_0^\lambda u^{-2}F(u)du\,, \tag{13}$$

which shows the useful monotonicity of the transform from F to G and H.

- The heat case (12) is clear since $A^{-1}e^{-tA}\Pi_V = \int_t^{+\infty} e^{-sA}\Pi_V ds$ by the spectral calculus.

□

The sequel of the proofs of Theorems 1.1, 1.2 and 1.3 relies on a classical technique from real interpolation theory, as used for instance in the elementary proof of the $L^2 - L^p$ Sobolev inequality in $\mathbb{R}^n$ given by Chemin and Xu in [5]. This consists here in estimating a level set $\{x, |f(x)| > y\}$ by using an appropriate spectral splitting of $f = \Pi_\lambda f + \Pi_{>\lambda} f$ for $f \in V$.

2.1. **Proof of Theorem 1.1.** By (2) and (11) one has $\|A^{-1/2}\Pi_\lambda\|_{2,\infty}^2 \leq G(\lambda)$, hence

$$\|\Pi_\lambda f\|_\infty^2 \leq G(\lambda)\|A^{1/2}f\|_2^2 = G(\lambda)\mathcal{E}(f)\,. \tag{14}$$

Then suppose that $|f(x)| \geq y$, with $y^2 = 4G(\lambda)\mathcal{E}(f)$. As $|\Pi_\lambda f(x)| \leq y/2$ by (14), one has necessarily $|\Pi_{>\lambda} f(x)| \geq y/2 \geq |\Pi_\lambda f(x)|$ and finally

$$|f(x)|^2 \leq 4|\Pi_{>\lambda} f(x)|^2 \quad \text{on} \quad \big\{x \in X \mid |f(x)|^2 \geq 4G(\lambda)\mathcal{E}(f)\big\}\,. \tag{15}$$

Hence a first integration in x gives,

$$\int_{\{x\,,\,|f(x)|^2 \geq 4\mathcal{E}(f)G(\lambda)\}} |f(x)|^2 d\mu \leq 4\|\Pi_{>\lambda} f\|_2^2\,,$$

and a second integration in λ,

$$\int_X \frac{|f(x)|^2}{4\mathcal{E}(f)} G^{-1}\Big(\frac{|f(x)|^2}{4\mathcal{E}(f)}\Big) d\mu(x) \leq \int_0^{+\infty} \frac{\|\Pi_{>\lambda} f\|_2^2}{\mathcal{E}(f)} d\lambda\,,$$

where $G^{-1}(y) = \sup\{\lambda \mid G(\lambda) \leq y\}$. At last the spectral calculus provides

$$\begin{aligned}\int_0^{+\infty} \|\Pi_{>\lambda} f\|_2^2\, d\lambda &= \int_0^{+\infty}\int_\lambda^{+\infty} \langle d\Pi_\mu f, f\rangle \\ &= \int_0^{+\infty} \mu\, \langle d\Pi_\mu f, f\rangle = \langle Af, f\rangle = \mathcal{E}(f)\,,\end{aligned}$$

giving Theorem 1.1.

2.2. **Proof of Theorem 1.2.** We follow the same lines as above. First by (2) and (12) one has for $f \in V$

$$\|e^{-tA/2} f\|_\infty \leq M(t)\mathcal{E}(f)\,,$$

leading to

$$|f(x)|^2 \leq 4|(1 - e^{-tA/2})f(x)|^2 \quad \text{on} \quad \big\{x \in X \mid |f(x)|^2 \geq 4M(t)\mathcal{E}(f)\big\}\,. \tag{16}$$

Then integrations in x and dt/t^2 give

$$\int_X \frac{|f(x)|^2}{4\mathcal{E}(f)} / M^{-1}\Big(\frac{|f(x)|^2}{4\mathcal{E}(f)}\Big)\, d\mu(x) \leq \frac{1}{\mathcal{E}(f)} \int_0^{+\infty} \|(1 - e^{-tA/2})f\|_2^2\, \frac{dt}{t^2}\,,$$

where now $M^{-1}(y) = \inf\{t \mid M(t) \geq y\}$ for the decreasing M. The right integral is computed by spectral calculus

$$\begin{aligned}\int_0^{+\infty} \|(1-e^{-tA/2})f\|_2^2 \,\frac{dt}{t^2} &= \int_0^{+\infty}\int_0^{+\infty} (1-e^{-t\lambda/2})^2 \langle d\Pi_\lambda f, f\rangle \,\frac{dt}{t^2} \\ &= \int_0^{+\infty}\Big(\int_0^{+\infty} \frac{(1-e^{-u})^2}{2u^2} du\Big) \lambda \langle d\Pi_\lambda f, f\rangle \\ &= I\mathcal{E}(f)\,,\end{aligned}$$

where $2I = \displaystyle\int_0^{+\infty} \frac{(1-e^{-u})^2}{u^2} du = 2\ln 2$ as seen developing $I_\varepsilon = \displaystyle\int_\varepsilon^{+\infty} \frac{(1-e^{-u})^2}{u^2} du$ when $\varepsilon \searrow 0$.

2.3. **Proof of Theorem 1.3.** Here one compares levels of f to $\|f\|_1$ instead of $\mathcal{E}(f)$. Using $F(\lambda) = \|\Pi_\lambda\|_{1,\infty}$ one gets

$$|f(x)|^2 \leq 4|\Pi_{>\lambda} f(x)|^2 \quad \text{on} \quad \big\{x \in X \mid |f(x)| \geq 2F(\lambda)\|f\|_1\big\}\,. \tag{17}$$

This leads to (5) by integration as before, from which follows the Nash–type inequality (6) by applying Jensen inequality to the convex function φ and the probability measure $dP = |f|\, d\mu/\|f\|_1$.

Remark 2.2. In the previous proofs, it appears clearly that the proposed controls of ultracontractive norms of spectral or heat decay are much stronger than the Sobolev and Nash-type inequalities deduced. Indeed these inequalities are twice integrated versions, in space and frequency, of the "local" inequalities (15), (16) and (17), that come directly from the ultracontractive controls. Therefore it seems hopeless to get the converse statements in general. However one can get back from Sobolev or Nash to heat decay, in the case the heat is equicontinuous on L^1; as due to Varopoulos in [15] for the polynomial case, and Coulhon in [6] for more general decays.

3. Relationships between inequalities

3.1. **From H-Sobolev to Nash.** We compare and comment briefly the various results obtained. At first, in the classical polynomial case, Sobolev inequality (1) implies Nash' one

$$\|f\|_2^{1+2/\alpha} \leq C\|f\|_1^{2/\alpha}\mathcal{E}(f)^{1/2}$$

by Hölder, see e.g. [7]. In the general case here one needs some convexity assumptions to get a Nash–type inequality from H or N-Sobolev.

Indeed, suppose either the H or N-Sobolev inequality (3) or (4) holds, and suppose φ is a convex function such that $\varphi(y) \leq yG^{-1}(y^2)$, resp. $\varphi(y) \leq y/M^{-1}(y^2)$. Then by Jensen the following Nash-type inequality is satisfied

$$\varphi\Big(\frac{\|f\|_2^2}{2\mathcal{E}(f)^{1/2}\|f\|_1}\Big) \leq \frac{2\mathcal{E}(f)^{1/2}}{\|f\|_1} \text{ resp. } \frac{2\ln 2\mathcal{E}(f)^{1/2}}{\|f\|_1}\,. \tag{18}$$

If one can take $\varphi(y) \geq CyG^{-1}(y^2)$ for some constant C, this leads to

$$\frac{\|f\|_2^2}{\|f\|_1^2} \leq \frac{4\mathcal{E}(f)}{\|f\|_2^2} G\Big(\frac{4\mathcal{E}(f)}{C\|f\|_2^2}\Big) \tag{19}$$

In comparison, the Nash inequality (6) provides

$$\frac{\|f\|_2^2}{2\|f\|_1^2} \leq F\Big(\frac{4\mathcal{E}(f)}{C\|f\|_2^2}\Big)\,, \tag{20}$$

if there exists a convex function ψ such that $CyF^{-1}(y) \leq \psi(y) \leq yF^{-1}(y)$. Up to constants this latter formula (20) is a priori sharper than (19), since $F(\lambda) \leq \lambda G(\lambda)$ in general.

Observe that one may have $F(\lambda) \ll \lambda G(\lambda)$ for very thick near-zero spectrum. For instance if $F(\lambda) = \lambda/\ln^2\lambda$ then $\lambda G(\lambda) = (-\ln\lambda + 1)F(\lambda)$. Except this "low dimensional" phenomenon, one has $\lambda G(\lambda) \underset{0}{\asymp} F(\lambda)$ in the other cases, and thus the two Nash inequalities (19) and (20) have same strength. For instance this holds if $F(\lambda) \underset{0}{\sim} \lambda^{1+\varepsilon}\varphi(\lambda)$ for some $\varepsilon > 0$ and an increasing $\varphi > 0$. This comes from the following remark.

Proposition 3.1. *Suppose there exists $\varepsilon > 0$ such that, for small λ, F satisfies the growing condition $F(2\lambda) \geq 2(1+\varepsilon)F(\lambda)$, then $(2+\varepsilon^{-1})F(\lambda) \geq \lambda G(\lambda) \geq F(\lambda)$.*

Proof. By (13), one has

$$\begin{aligned} G(\lambda) &= \int_0^\lambda \frac{dF(u)}{u} = \frac{F(\lambda)}{\lambda} + \int_0^\lambda \frac{F(u)}{u^2}du \\ &= \frac{F(\lambda)}{\lambda} + \Big(\int_0^{\lambda/2} + \int_{\lambda/2}^{\lambda}\Big)\frac{F(u)}{u^2}du \\ &\leq \frac{2F(\lambda)}{\lambda} + \int_0^{\lambda/2} \frac{F(2u)}{2(1+\varepsilon)u^2}du \quad \text{by hypothesis on } F\,, \\ &\leq \frac{2F(\lambda)}{\lambda} + \frac{1}{1+\varepsilon}\Big(G(\lambda) - \frac{F(\lambda)}{\lambda}\Big)\,, \end{aligned}$$

leading to $\lambda G(\lambda) \leq (2+\varepsilon^{-1})F(\lambda)\,.$ □

As a curiosity, we note that under the growing hypothesis on F above, the spectral density of states F and the spatial repartition function H have symmetric expressions with respect to G and G^{-1}. Indeed, one has simply there

$$F(\lambda) \asymp \lambda G(\lambda) \quad \text{while} \quad H(x) = xG^{-1}(x)\,. \tag{21}$$

3.2. **Spectral versus heat decay.** One would like to compare the two Theorems 1.1 and 1.2. They both lead to Sobolev inequalities starting either from the heat or spectral decay. One can compare F and G to L and M through Laplace transform of associated measures.

Proposition 3.2. • *In any case it holds that*

$$L(t) \leq \mathcal{L}(dF)(t) = \int_0^{+\infty} e^{-\lambda t} dF(\lambda) \tag{22}$$

$$M(t) \leq \mathcal{L}(dG)(t) = \int_0^{+\infty} e^{-\lambda t} dG(\lambda)\,. \tag{23}$$

• *If A is an invariant operator acting on L^2-sections of an invariant vector bundle V over a locally compact group Γ, then reverse inequalities hold up to the multiplicative factor*

$n = \dim V$, *i.e.*

$$\mathcal{L}(dF) \leq nL \quad \text{and} \quad \mathcal{L}(dG) \leq nM\,.$$

Moreover $G(y) \leq neM(y^{-1})$ *and H-Sobolev inequality* (3) *implies N-Sobolev* (4)*, up to multiplicative constants.*

- *Reversely, for any operator, if* G *satisfies the exponential growing condition :*

$$\exists C \text{ such that } \forall u, y > 0\,,\ G(uy) \leq e^{Cu} G(y)\,,$$

then $M(y^{-1}) \leq 3G(2Cy)$*. Hence* H *and* N*-Sobolev are equivalent on groups in that case.*

Proof. • By spectral calculus $e^{-tA}\Pi_V = \int_0^{+\infty} e^{-t\lambda} d\Pi_\lambda = t\int_0^{+\infty} e^{-t\lambda}\Pi_\lambda d\lambda$, hence

$$L(t) = \|e^{-tA}\Pi_V\|_{1,\infty} \leq t\int_0^{+\infty} e^{-t\lambda}\|\Pi_\lambda\|_{1,\infty} d\lambda = \mathcal{L}(dF)(t)\,,$$

and thus

$$M(t) = \int_t^{+\infty} L(s)ds \leq \int_t^{+\infty}\int_0^{+\infty} e^{-\lambda s} dF(\lambda) ds = \int_0^{+\infty} \frac{e^{-\lambda t}}{\lambda} dF(\lambda) = \mathcal{L}(dG)(t)\,.$$

• For positive invariant operators P on groups, we will see in Proposition 4.2 that the ultracontractive norm $\|P\|_{1,\infty}$ is pinched between the trace $\tau_\Gamma(P)$ and $n\tau_\Gamma(P)$. This gives the reverse inequalities by linearity of τ_Γ. In particular one gets

$$\begin{aligned} nM(y^{-1}) &\geq \int_0^{+\infty} e^{-\lambda/y} dG(\lambda) = y^{-1}\int_0^{+\infty} e^{-\lambda/y} G(\lambda) d\lambda \\ &\geq y^{-1}\int_y^{+\infty} e^{-\lambda/y} G(y) d\lambda = e^{-1}G(y). \end{aligned}$$

Therefore $N(y) = y/M^{-1}(y^{-1}) \leq yG^{-1}(ey) = e^{-1}H(ey)$ and H-Sobolev implies

$$\int_X N\Big(\frac{|f(x)|^2}{4e\mathcal{E}(f)}\Big) d\mu \leq e^{-1}\,.$$

• If G satisfies the growing condition, one has by (23)

$$\begin{aligned} M(1/y) &\leq \int_0^{+\infty} e^{-\lambda/y} dG(\lambda) = \int_0^{+\infty} e^{-u} G(uy) du \\ &\leq \int_0^{2C} e^{-u} G(2Cy) du + \int_{2C}^{+\infty} e^{-u/2} G(2Cy) du \\ &\leq 3G(2Cy)\,. \end{aligned}$$

□

We note that it may happen that $N \ll H$ for very thin near–zero spectrum. In an extreme case there may be a gap in the spectrum, i.e. $A \geq \lambda_0 > 0$, hence $F = G = H = 0$ near zero, while $L(t) \asymp Ce^{-ct}$, $M(t) \asymp C'e^{-ct}$ and $N(y) \asymp C''y/\ln(y/C')$.

4. Ultracontractive norms and Γ-trace.

For applications we now discuss some geometric aspect of the analytic spectral decay $F(\lambda) = \|\Pi_\lambda\|_{1,\infty}$ we consider.

In the case of operators invariant under the action of a group Γ, such hypercontractive norms are related to von Neumann Γ-dimension and trace. We briefly recall these notions and refer for instance to [13, §2] for more details. However we will follow here a slightly different approach, as in [14, §6.1] for instance, that covers also some non-discrete actions.

Suppose that a locally compact group Γ (discrete or not) acts by measure preserving transforms on the space X with a *finite* quotient X/Γ. For instance, when Γ is discrete, X may be a covering space over a finite simplicial complex. Equivalently one can also take a d–dimensional invariant bundle V over a group Γ and set $X = \Gamma \times [1, d]$, so that $L^2(X) \simeq L^2(\Gamma) \otimes V_e$.

The following straightforward proposition, see e.g. [14, Prop. 6.4–6.6], leads to a definition of a "Γ-trace" in this setting.

Proposition 4.1. *Let Γ be a locally compact group and P be a Γ-invariant positive operator on $L^2(\Gamma) \otimes V_e$. For any $D \subset \Gamma$ with Haar measure $0 < \lambda(D) < +\infty$, consider the trace*

$$\tau_D(P) = \lambda(D)^{-1} \operatorname{Tr}(\chi_D P \chi_D)\,.$$

• *Let S be the positive square root of P. Then $\tau_D(P)$ is finite iff $S\chi_D$ is an Hilbert–Schmidt operator. In that case the kernel of S is $K_S(x, y) = k_S(y^{-1}x)$ with $k_S \in L^2(\Gamma)$, while the kernel of P is $K_P(x, y) = k_P(y^{-1}x)$ with $k_P = k_S * k_S \in C_0(\Gamma)$, and it holds that*

$$\tau_D(P) = \int_\Gamma \operatorname{Tr}_{V_e}\big(k_S^*(x) k_S(x)\big) d\lambda(x) = \operatorname{Tr}_{V_e}(k_P(e))\,.$$

In particular this trace is independent of D. It will be denoted by τ_Γ and called (improperly) the Γ–trace in the sequel.

• *If moreover Γ is* unimodular, *and P is a (not necessarily positive) Γ–invariant bounded operator, then $\tau_\Gamma(P^*P) = \tau_\Gamma(PP^*)$. Hence τ_Γ actually defines a faithful trace in that case.*

We recall that this last trace property allows to get a meaningful notion of dimension for closed Γ-invariant subspaces $L \subset H = L^2(\Gamma) \otimes V_e$. Indeed, one sets then $\dim_\Gamma L = \operatorname{Tr}_\Gamma(\Pi_L)$. This satisfies the key property $\dim_\Gamma f(L) = \dim_\Gamma L$ for any closed densely defined invariant injective operator $f : L \to H$, see e.g. [13, §2] or [14, §3.2].

On any locally compact group, the Γ-trace of P is easily compared to its ultracontractive norm.

Proposition 4.2. *Let P be a positive Γ-invariant operator acting on $L^2(X) = L^2(\Gamma) \otimes V_e$ with kernel $K_P(x, y) = k_P(y^{-1}x)$, then*

$$\|P\|_{1,\infty} = \|k_P(e)\| \le \tau_\Gamma(P) \le (\dim V_e)\|P\|_{1,\infty}\,.$$

Proof. In general one has $\|P\|_{1,\infty} = \sup_{x,y} \|K_P(x, y)\|$, and by positivity of P,

$$2|\langle K_P(x, y)u, v\rangle| \le \langle K_P(x, x)u, u\rangle + \langle K_P(y, y)v, v\rangle\,.$$

Therefore $\|P\|_{1,\infty} = \sup_x \|K_P(x,x)\| = \|k_P(e)\|$ for an invariant operator. Here

$$\|k_P(e)\| = \sup_{\|v\|\leq 1} \|k_P(e)v\|_{V_e} = \sup_{\|v\|\leq 1} \langle k_P(e)v, v\rangle$$

for the positive $k_P(e)$, while $\tau_\Gamma(P) = \mathrm{Tr}_{V_e}(k_P(e))$ by Proposition 4.1. □

As a consequence, already used in Proposition 3.2, the norm $\|P\|_{1,\infty}$ is, up to multiplicative constants, a linear form on positive P. This gives also the converse inequalities to (11) and (12) in Proposition 2.1 for invariant operators on groups. Indeed it holds in this case that

$$\begin{aligned} \|A^{-1}\Pi_\lambda\|_{1,\infty} &\asymp \tau_\Gamma(A^{-1}\Pi_\lambda) \asymp G(\lambda) \\ \|A^{-1}e^{-tA}\|_{1,\infty} &\asymp \tau_\Gamma(A^{-1}e^{-tA}) \asymp M(t)\,, \end{aligned} \tag{24}$$

due to the equalities $\tau_\Gamma(A^{-1}e^{-tA}) = \int_t^{+\infty} \tau_\Gamma(e^{-sA})ds$ and

$$\tau_\Gamma(A^{-1}\Pi_\lambda) = \int_0^\lambda u^{-1} d\tau_\Gamma(\Pi_u) = \lambda^{-1}\tau_\Gamma(\Pi_\Lambda) + \int_0^\lambda u^{-2}\tau_\Gamma(\Pi_u)du\,.$$

Its relation to the Γ-trace allows to estimate the ultracontractive spectral decay $F(\lambda)$ of A in some simple cases. Namely, following Dixmier [8, §18.8], if the group Γ is locally compact unimodular and postliminaire, there exists a Plancherel measure μ on its unitary dual $\widehat{\Gamma}$, together with a Plancherel formula that gives here

$$F(\lambda) = \|\Pi_\lambda\|_{1,\infty} \asymp \tau_\Gamma(\Pi_\lambda) = \int_{\widehat{G}} \mathrm{Tr}(\widehat{\Pi_\lambda}(\xi))d\mu(\xi)\,. \tag{25}$$

For instance, in the case of the Laplacian Δ on $\mathbb{R}^n$, the spectral space $E_\lambda(\Delta)$ is the Fourier transform of functions supported in the ball $B(0,\sqrt{\lambda})$ in $(\widehat{\mathbb{R}^n}, d\mu) \simeq (\mathbb{R}^n, (2\pi)^{-n}dx)$, hence

$$F(\lambda) = \mu(B(0,\sqrt{\lambda})) = C_n\lambda^{n/2},$$

with $C_n = (2\pi)^{-n}\mathrm{vol}(B_n)$. This leads to

$$G(\lambda) = \frac{nC_n}{n-2}\lambda^{n/2-1} \quad \text{and} \quad H(x) = xG^{-1}(x) = \Big(\frac{n-2}{nC_n}\Big)^{\frac{2}{n-2}} x^{\frac{n}{n-2}}\,,$$

so that finally (3) gives the classical Sobolev inequality in $\mathbb{R}^n$

$$\|f\|_{2n/(n-2)} \leq \frac{1}{\pi}\Big(\frac{n\,\mathrm{vol}(B_n)}{n-2}\Big)^{\frac{1}{n}}\|df\|_2\,.$$

Yet we recall that the best constant here is $2(n(n-2))^{-1/2}\mathrm{area}(S^n)^{-1/n}$, see [2].

Still on $\mathbb{R}^n$, one can get some general algebraic expression of $F(\lambda)$ for positive invariant differential operator $A = \sum_I a_I\partial_{x_I}$. Let $\sigma(A)(\xi) = \sum_I a_I(i\xi)^I$ be its polynomial symbol. Then again the spectral space $E_\lambda(A)$ consists in functions whose Fourier transform is supported in

$$D_\lambda = \{\xi \in \mathbb{R}^n \mid \sigma(A)(\xi) \leq \lambda\}$$

and

$$F(\lambda) = (2\pi)^{-n}\mathrm{vol}(D_\lambda).$$

The asymptotic behaviour of $F(\lambda)$ when $\lambda \searrow 0$ can be obtained from the resolution of the singularity of the polynomial $\sigma(A)$ at 0. Indeed, there exists $\alpha \in \mathbb{Q}^+$ and $k \in [0, n-1] \cap \mathbb{N}$ such that

$$F(\lambda) \underset{\lambda \to 0^+}{\sim} C\lambda^\alpha |\ln \lambda|^k ,$$

see e.g. Theorem 7 in [1, §21.6]. Moreover, under a non-degeneracy hypothesis on $\sigma(A)$, the exponents α and k can be read from its Newton polyhedra. Then if $\alpha > 1$, Proposition 3.1 yields that $G(\lambda) \underset{0}{\asymp} \lambda^{\alpha-1}|\ln \lambda|^k$. Therefore $G^{-1}(u) \underset{0}{\asymp} u^{1/(\alpha-1)}|\ln u|^{-k/(\alpha-1)}$ and finally the H-Sobolev inequality (3) is governed in small energy by the function

$$H(u) \asymp u^{\frac{\alpha}{\alpha-1}} |\ln(u)|^{-\frac{k}{\alpha-1}} \quad \text{for} \quad u \ll 1 .$$

5. Spectral density and cohomology

To apply the previous results, we suppose now that K is a finite simplicial complex and consider a covering $\Gamma \to X \to K$. Let d_k be the coboundary operator on k-cochains X^k of X. As a purely combinatorial and local operator, it acts boundedly on all ℓ^p-spaces of cochains $\ell^p X^k$, see e.g. [3, 13].

Let $F_{\Gamma,k}(\lambda)$ denotes the Γ-trace of the spectral projector $\Pi_\lambda = \chi(]0, \lambda])$ of $A = d_k^* d_k$. By Proposition 4.2 this function is equivalent, up to multiplicative constants, to the hypercontractive spectral decay $F(\lambda) = \|\Pi_\lambda\|_{1,\infty}$. Thus Theorem 1.4 is a direct application of Theorem 1.1 in the polynomial case. This statement compares two measurements of the torsion of ℓ^2-cohomology $T_2^{k+1} = \overline{d_k(\ell^2)}^{\ell^2} / d_k(\ell^2)$ that share some geometric invariance. We describe this more precisely.

We first recall the main invariance property of $F_{\Gamma,k}(\lambda)$. We say that two increasing functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ are equivalent if there exists $C \geq 1$ such that $f(\lambda/C) \leq g(\lambda) \leq f(C\lambda)$ for λ small enough. According to [9, 12, 11] we have :

Theorem 5.1. *Let K be a finite simplicial complex and $\Gamma \to X \to K$ a covering. Then the equivalence class of $F_{\Gamma,k}$ only depends on Γ and the homotopy class of the $(k+1)$-skeleton of K.*

One tool in the proof is the observation that an homotopy of finite simplicial complexes F and G induces bounded Γ-invariant homotopies between the Hilbert complexes $(\ell^2 X^k, d_k)$ and $(\ell^2 Y^k, d'_k)$. That means there exist Γ-invariant bounded maps

$$f_k : \ell^2 X^k \to \ell^2 Y^k \quad \text{and} \quad g_k : \ell^2 Y^k \to \ell^2 X^k$$

such that

$$f_{k+1} d_k = d'_k f_k \quad \text{and} \quad g_{k+1} d'_k = d_k g_k$$

and

$$g_k f_k = \mathrm{Id} + d_{k-1} h_k + h_{k+1} d_k \quad \text{and} \quad f_k g_k = \mathrm{Id} + d'_{k-1} h'_k + h'_{k+1} d'_k$$

for some bounded maps

$$h_k : \ell^2 X^k \to \ell^2 X^{k-1} \quad \text{and} \quad h'_k : \ell^2 Y^k \to \ell^2 Y^{k-1}.$$

All these maps are purely combinatorial and local, see e.g. [3, 14], and thus extend on all ℓ^p spaces of cochains.

One can show a similar invariance property of the inclusion (9) we recall below, but that holds more generally on uniformly *locally finite simplicial complexes*, without requiring a group invariance. These are simplicial complexes such that each point lies in a bounded number $N(k)$ of k-simplexes.

Proposition 5.2. *Let X and Y be uniformly locally finite simplicial complexes. Suppose that they are boundedly homotopic in ℓ^2 and ℓ^p norms for some $p \geq 2$. Then one has*

$$\overline{d_k(\ell^2 X^k)}^{\ell^2} \subset d_k(\ell^p X^k)\,, \tag{9}$$

if and only if a similar inclusion holds on Y.

Proof. Suppose that $\overline{d_k(\ell^2 X^k)}^{\ell^2} \subset d_k(\ell^p X^k)$ and consider a sequence $\alpha_n = d'_k(\beta_n) \in d'_k(\ell^2 Y^k)$ that converges to $\alpha \in \overline{d_k(\ell^2 Y^k)}^{\ell^2}$ in ℓ^2.

Then $g_{k+1}\alpha_n = d_k(g_k\beta_n) \to g_{k+1}\alpha \in \overline{d_k(\ell^2 X^k)}^{\ell^2}$. Therefore there exists $\beta \in \ell^p X^k$ such that $g_{k+1}\alpha = d_k\beta$. Then taking ℓ^2-limit in the sequence

$$f_{k+1}g_{k+1}\alpha_n = \alpha_n + d'_k h'_{k+1}\alpha_n + h'_{k+2}d'_{k+1}\alpha_n = \alpha_n + d'_k h'_{k+1}\alpha_n$$

gives

$$d'_k(f_k\beta) = f_{k+1}d_k\beta = \alpha + d'_k h'_{k+1}\alpha\,,$$

and finally $\alpha \in d'_k(\ell^p Y^k)$ since $\ell^2 Y^k \subset \ell^p Y^k$ for $p \geq 2$. □

The inclusion (9) we consider here is related to problems studied in $\ell^{p,q}$ cohomology. We briefly recall this notion and refer for instance to [10] for details. If X is a simplicial complex as above, one considers the spaces

$$Z^k_q(X) = \ker d_k \cap \ell^q X^k \quad \text{and} \quad B^k_{p,q}(X) = d_{k-1}(\ell^p X^k) \cap \ell^q X^k\,.$$

Then the $\ell^{p,q}$-cohomology of X is defined by

$$H^k_{p,q}(X) = Z^k_q(X)/B^k_{p,q}(X)\,.$$

Its reduced part is the Banach space

$$\overline{H}^k_{p,q}(X) = Z^k_q(X)/\overline{B}^k_{p,q}(X)\,,$$

while its torsion part

$$T^k_{p,q}(X) = \overline{B}^k_{p,q}(X)/B^k_{p,q}(X)$$

is not a Banach space. These spaces fit into the exact sequence

$$0 \to T^k_{p,q}(X) \to H^k_{p,q}(X) \to \overline{H}^k_{p,q}(X) \to 0\,.$$

It is straightforward to check as above that, for $p \geq q$, these spaces satisfy the same homotopical invariance property as in Proposition 5.2.

Proposition 5.3. *Let X and Y be uniformly locally finite simplicial complexes. Suppose that they are boundedly homotopic in ℓ^p and ℓ^q norms for $p \geq q$. Then the maps $f_k : \ell^* X^k \to \ell^* Y^k$ and $g_k : \ell^* Y^k \to \ell^* X^k$ induce reciprocal isomorphisms between the $\ell^{p,q}$ cohomologies of X and Y, as well as their reduced and torsion components.*

In this setting, the vanishing of the $\ell^{p,2}$-torsion $T^{k+1}_{p,2}(X)$ is equivalent to the closeness of $B^{k+1}_{p,2}(X) = d_k(\ell^p X^k) \cap \ell^2 X^{k+1}$ in $\ell^2 X^{k+1}$, i.e to the inclusion

$$\overline{d_k(\ell^p X^k) \cap \ell^2 X^{k+1}}^{\ell^2} \subset d_k(\ell^p X^k) \cap \ell^2 X^{k+1} .$$

This implies the weaker inclusion (9), but is stronger in general unless the following holds

$$d_k(\ell^p X^k) \cap \ell^2 X^{k+1} \subset \overline{d_k(\ell^2 X^k)}^{\ell^2} . \tag{26}$$

Now by Hodge decomposition in $\ell^2 X^{k+1}$, one has always

$$d_k(\ell^p X^k) \cap \ell^2 X^{k+1} \subset \ker d_{k+1} \cap \ell^2 X^{k+1} = \overline{H}^{k+1}_2(X) \oplus^{\perp} \overline{d_k(\ell^2 X^k)}^{\ell^2} .$$

Hence (26) holds if the reduced ℓ^2-cohomology $\overline{H}^{k+1}_2(X)$ vanishes, proving in that case the equivalence of (9) to the vanishing of the $\ell^{p,2}$-torsion, and even to the identity

$$B^{k+1}_{p,2} := d_k(\ell^p X^k) \cap \ell^2 X^{k+1} = \overline{d_k(\ell^2 X^k)}^{\ell^2} , \tag{27}$$

which is clearly closed in ℓ^2.

Corollary 5.4. *Let K be a finite simplicial space and $\Gamma \to X \to K$ a covering. Suppose that the spectral distribution $F_{\Gamma,k}$ of $A = d_k^* d_k$ on $(\ker d_k)^{\perp}$ satisfies $F_{\Gamma,k}(\lambda) \leq C\lambda^{\alpha/2}$ for some $\alpha > 2$. Suppose moreover that the reduced ℓ^2-cohomology $\overline{H}^{k+1}_2(X)$ vanishes.*

Then (27) and the vanishing of the $\ell^{p,2}$-torsion $T^{k+1}_{p,2}(X)$ hold for $1/p \leq 1/2 - 1/\alpha$.

For instance, by [4], infinite amenable groups have vanishing reduced ℓ^2-cohomology in all degrees.

References

[1] V. I. Arnol′d, S. M. Guseĭn-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Vol. II*, volume 83 of *Monographs in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1988. Monodromy and asymptotics of integrals, Translated from the Russian by Hugh Porteous, Translation revised by the authors and James Montaldi.

[2] T. Aubin. *Nonlinear analysis on manifolds. Monge-Ampère equations*, volume 252 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1982.

[3] M. Bourdon and H. Pajot. Cohomologie l_p et espaces de Besov. *J. Reine Angew. Math.*, 558:85–108, 2003.

[4] J. Cheeger and M. Gromov. L^2-cohomology and group cohomology. *Topology*, 25(2):189–215, 1986.

[5] J.-Y. Chemin and C.-J. Xu. Inclusions de Sobolev en calcul de Weyl-Hörmander et champs de vecteurs sous-elliptiques. *Ann. Sci. École Norm. Sup. (4)*, 30(6):719–751, 1997.

[6] T. Coulhon. Ultracontractivity and Nash type inequalities. *J. Funct. Anal.*, 141(2):510–539, 1996.

[7] T. Coulhon. Heat kernel and isoperimetry on non-compact Riemannian manifolds. In *Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002)*, volume 338 of *Contemp. Math.*, pages 65–99. Amer. Math. Soc., Providence, RI, 2003.

[8] J. Dixmier. *Les C^*-algèbres et leurs représentations*. Deuxième édition. Cahiers Scientifiques, Fasc. XXIX. Gauthier-Villars Éditeur, Paris, 1969.

[9] A. V. Efremov. Cell decompositions and the Novikov-Shubin invariants. *Uspekhi Mat. Nauk*, 46(3(279)):189–190, 1991.

[10] V. Gol′dshtein and M. Troyanov. Sobolev inequalities for differential forms and $L_{q,p}$-cohomology. *J. Geom. Anal.*, 16(4):597–631, 2006.

[11] M. Gromov and M. Shubin. Erratum to: "von Neumann spectra near zero" [Geom. Funct. Anal. **1** (1991), no. 4, 375–404; MR 92i:58184]. *Geom. Funct. Anal.*, 5(4):729, 1995.

[12] M. Gromov and M. A. Shubin. von Neumann spectra near zero. *Geom. Funct. Anal.*, 1(4):375–404, 1991.

[13] P. Pansu. Introduction to L^2 Betti numbers. In *Riemannian geometry (Waterloo, ON, 1993)*, pages 53–86. Amer. Math. Soc., Providence, RI, 1996.

[14] M. Rumin. An introduction to spectral and differential geometry in Carnot-Carathéodory spaces. *Rend. Circ. Mat. Palermo (2) Suppl.*, (75):139–196, 2005.

[15] N. T. Varopoulos, L. Saloff-Coste, and T. Coulhon. *Analysis and geometry on groups*. Cambridge University Press, Cambridge, 1992.

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